

Cyclic hybrid systems of flows on manifolds *

Witold Szczechla

Abstract

The considered continuous-and-discrete hybrid system is a cyclic relay of smooth flows on an n -dimensional manifold M , where the discrete process of switching from each flow to the next takes place on the boundaries of some fixed compact n -dimensional submanifolds of M . The main result is the existence of at least one periodic cyclic trajectory under some topological condition concerning one of the submanifold boundaries.

1 Introduction

Whereas single vector fields on manifolds and their local combinations have been extensively studied, especially in the areas of theoretical mechanics, dynamical systems and differential geometry, the complexity of the *global behaviour of combined trajectories* has attracted less research, even in control-theoretic setting. Naturally arising in this context are relays of flows on manifolds, defined as special smooth-and-discrete hybrid systems in [14]. They constitute a special case of nonlinear systems studied, in particular, in [11], [14], [19], [16], [17].

A *relay of flows* is a system whose phase space is a smooth (C^∞) manifold M of dimension $n > 0$. Let p be an integer, $p > 1$. One is given a cyclic collection of smooth flows on M ,

$$F_k : M \times \mathbf{R} \rightarrow M \quad (k \in \mathbf{Z}_p),$$

and a corresponding collection of compact n -dimensional manifolds M_k with nonempty boundaries ($k \in \mathbf{Z}_p$). The following two conditions are assumed:

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(i) For each k we have

$$\partial M_{k-1} \cap M_k = \emptyset.$$

(ii) For each k there is a value $T_k > 0$ such that

$$F_k(T_k, \partial M_{k-1}) \subset \text{int} M_k.$$

The cyclic hybrid system thus obtained will be called \mathcal{S} . A *trajectory* (i.e., a *solution* or more precisely, a *quasisolution*) of the system \mathcal{S} is defined to be a pair of functions (F, α) with $F : [t_0, \infty) \rightarrow M$ continuous and $\alpha : [t_0, \infty) \rightarrow \mathbf{Z}_p$ piecewise constant. The following two conditions must be satisfied:

(i) If $\alpha|_{[a, b]} = k = \text{const}$, then

$$F(b) = F_k^{b-a}(F(a)).$$

(ii) If $\alpha(t-0) = k \neq \alpha(t+0)$, then $\alpha(t+0) = k+1$ and $F(t) \in \partial M_k$.

Note. One speaks of ‘quasisolutions’ since the term ‘solution’ is sometimes reserved for the trajectories satisfying the following additional condition, not considered in this paper: (iii) if $\alpha(t) = k$, then $F(t) \notin \text{int} M_k$.

Motivation. Besides its theoretical interest, the model has been motivated by the switching occurring in electromagnetic fields and/or some designs of control systems engineering.

1.1 The limit sets for solutions

For any initial point $x_0 \in M$ and initial mode $\alpha(0+) = k_0$ we define the accessible set $A(x_0, k_0)$ and the ω -limit set $\Omega(x_0, k_0)$ for *solutions* of \mathcal{S} in the usual way (by analogy to dynamical systems, cf. [5]). We note that trajectories can be nonunique and any point in the ω -limit set has (by definition) to be approached by *some* trajectory starting at the end of *every* trajectory of x_0 which has just completed m switches (where m is arbitrary).

Proposition 1 *Each of the sets $A(x_0, k_0)$ and $\Omega(x_0, k_0)$ is connected.*

Proof. The set $A(x_0, k_0)$ is made up of consecutive pieces of trajectories between ∂M_{k-1} and ∂M_k which may be branching, but connect at the ends. Hence, the closure of $A(x_0, k_0)$ is always connected. By definition, the limit set $\Omega(x_0, k_0)$ can be expressed as the intersection of a descending sequence of such closures. ■

2 Periodic quasisolutions

In general, the system \mathcal{S} may have no periodic quasisolution. Such may be the case if $p = 2$, $M = S^2$, and the M_k are two disjoint disks whose boundaries are twisted by the first return map. In such a case, however, the disk M_0 can never be entirely carried inside M_1 by the flow F_1 because of an argument using Brouwer fixed point theorem. In fact, the following specific property of the sphere S^{n-1} , related to the degree (mod 2) of a mapping, turns out to be relevant to our method.

Definition 1 Let S be a compact manifold. We will say that S has the *coincidence property (mod 2)* if S is connected and if for every compact manifold N of the same dimension, for every pair of smooth maps $f, g : N \rightarrow S$ satisfying the condition $\deg_2(f) \neq \deg_2(g)$, there exists a point of coincidence (i.e., a point $x \in N$ such that $f(x) = g(x)$).

Let us notice that the sphere of any dimension, $S = S^{n-1}$, has the above property. Indeed, suppose $f(x) \neq g(x)$ for all $x \in N$. Then f is homotopic to the map $-g = \sigma \circ g$, where σ is the antipodal map. However, since $\deg_2(\sigma) = 1$, this implies $\deg_2(f) = \deg_2(-g) = 1 \cdot \deg_2(g)$.

Theorem 1 Suppose that, for some index $\beta \in \mathbf{Z}_p$, we have

$$F_\beta(t, M_{\beta-1}) \subset \text{int} M_\beta \quad \text{for all } t \geq T_\beta. \quad (1)$$

Suppose also that, for some index $k \in \mathbf{Z}_p$, the boundary ∂M_k has the coincidence property (mod 2). Then the system \mathcal{S} has a p -periodic quasisolution.

We may, and will, assume that the manifold ∂M_0 has the coincidence property (mod 2). Let $x \in \partial M_{k-1}$, where k is arbitrary. By the assumptions (i) and (ii), it is reasonable to suppose that, generically, there are an odd

number of values $t \in (0, T_k)$ for which $F_k(t, x) \in \partial M_k$, since $x \notin M_k$ and $F_k(T_k, x) \in \text{int} M_k$. On the other hand, by assumption (1), the *backward* trajectory of the flow F_β starting at a generic point $x \in \partial M_\beta$ should cross $\partial M_{\beta-1}$ an even number of times as neither x nor $F_\beta(-T_k, x)$ can belong to $M_{\beta-1}$. We want to relate the above observations to the degree mod 2 of certain maps into ∂M_0 .

2.1 Technical lemmas

For technical reasons, we will approximate our system of submanifolds by a certain family of similar systems continuously parametrized by $\lambda \in \mathbf{R}^{p+1}$ and prove a suitably generalized theorem for a residual set of parameters. The new system will consist of manifolds M_i^λ ($i = 0, \dots, p$), where M_i^λ is close to M_i , but possibly $M_0^\lambda \neq M_p^\lambda$. In particular, i is not a cyclic index here. The flows will be unchanged and indexed 1 to p .

For $i = 0, 1, \dots, p$, pick a smooth function $f_i : M \rightarrow \mathbf{R}$ having the following properties:

- (1) $f_i^{-1}(\mathbf{R}_+) = M_i$
- (2) 0 is a regular value of f_i .
- (3) If $\lim_{n \rightarrow \infty} x_n = 0$ then $\lim_{n \rightarrow \infty} \text{dist}(x_n, \partial M_i) = 0$.

We note that, by (2) and (3), every value sufficiently close to 0 is regular.

Lemma 1 *There exist functions f_i satisfying (1)–(3).*

Proof. First we take flows Ψ_i , transverse to ∂M_i and directed outwards. Next, define the f_i on neighborhoods of ∂M_i by

$$f_i(0)\Psi_i(t, \xi) = -\nu(t), \quad (\xi \in \partial M_i, |t| < \epsilon), \quad (2)$$

where ϵ is so small that the maps $(t, \xi) \mapsto \Psi_i(t, \xi)$ are diffeomorphic embeddings of $(-\epsilon, \epsilon) \times \partial M_i$ into M and

$$\nu(t) = \begin{cases} t & \text{for } |t| < \epsilon/4 \\ \epsilon/3 & \text{for } |t| > \epsilon/2. \end{cases}$$

Now it is enough to extend each f_i by $\epsilon/3$ on M_i and $-\epsilon/3$ on $M \setminus M_i$. ■

For $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_p) \in \mathbf{R}^{p+1}$, denote $f_i^{-1}(0)[\lambda_i, \infty)$ by M_i^λ . Let $\Lambda \subset \mathbf{R}^{p+1}$ be the set of those $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_p)$ which satisfy the following conditions:

- (4) For each i , λ_i is a regular value of f_i
- (5) The sets M_i^λ satisfy the hypothesis of our theorem with M_i^λ playing the role of M_i except possibly for the requirement $M_0^\lambda = M_p^\lambda$.
- (6) $F_i(T_i, \partial M_{i-1}^\lambda) \subset \text{int} M_i^\lambda$ ($i = 1, \dots, p$) and $F_\beta(T_\beta, M_{\beta-1}^\lambda) \subset \text{int} M_\beta^\lambda$.

Using properties (1)–(3), it is easy to show that Λ is an open set and that $0 \in \text{cl } \Lambda$. For the remainder of the proof it will be usually understood that $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_p) \in \Lambda$. A generic property means a property holding on a residual subset of Λ .

2.2 The manifold of the switching points

Let us define the set $N^\lambda \subset M \times \mathbf{R}^p$ to be the set of the points (x, t_1, \dots, t_p) satisfying:

- (7) $0 < t_i < T_i$ for each i ;
- (8) if we denote $x_0 = x$ and $x_{i+1} = F_{i+1}(t_{i+1}, x_i)$, then $x_i \in \partial M_i^\lambda$ for each i .

Notice that N^λ is always nonempty and compact. (While we do admit the possibility of N^λ being disconnected, the essential assumption is the connectivity of ∂M_0 .)

Lemma 2 *Generically, N^λ is a smooth $(n-1)$ -dimensional submanifold of $M \times \mathbf{R}^p$.*

Proof. Let $\Omega = M \times (0, T_1) \times \dots \times (0, T_p)$. Define the map $\nu : \Omega \rightarrow \mathbf{R}^{p+1}$ in the following way. Let $\omega = (x, t_1, \dots, t_p) \in \Omega$, $x_0 = x$ and $x_{i+1} = F_{i+1}(t_{i+1}, x_i)$ ($i = 0, \dots, p-1$); then

$$\nu(\omega) = (0)f_0(x_0), f_1(x_1), \dots, f_p(x_p).$$

Then $N^\lambda = \nu^{-1}(\lambda)$, so N^λ is a smooth submanifold of dimension $\dim \Omega - \dim \mathbf{R}^{p+1} = n - 1$ whenever λ is a regular value of ν . By Morse-Sard theorem, the set of the regular values is residual in \mathbf{R}^{p+1} and hence in Λ . ■

2.3 Computation of the degrees mod 2

Define the maps $\nu_0 : N^\lambda \rightarrow \partial M_0^\lambda$ and $\nu_1 : N^\lambda \rightarrow \partial M_p^\lambda$ by

$$\nu_0(x, t_1 \dots t_p) = x$$

and

$$\nu_1(x, t_1 \dots t_p) = F_p^{t_p} \circ \dots \circ F_1^{t_1}(x),$$

where we used the notation $F_i^t(z) = F_i(t, z)$.

Notice that Lemma 2, together with the connectivity of the manifolds ∂M_0 and ∂M_p^λ , allow us to compute, for almost every λ , the degree modulo 2 of ν_0 and ν_1 . For any $x \in M$ we put $R_i^i(x) = \{x\}$ and define the sets $R_i^j(x)$ ($i, j = 0, \dots, p$) recursively by

$$R_i^{j+1}(x) = \bigcup_{0 < t < T_{j+1}} F_{j+1}^t(0) R_i^j(x) \cap \partial M_{j+1}^\lambda \quad \text{for } i \leq j < p$$

and

$$R_i^{j-1}(x) = \bigcup_{0 < t < T_j} F_j^{-t}(0) R_i^j(x) \cap \partial M_{j-1}^\lambda \quad \text{for } 0 < j \leq i.$$

Also, let V_i denote the vector field inducing the flow F_i .

Lemma 3 *Generically, if N^λ is a smooth $(n - 1)$ -dimensional submanifold, then $\deg_2 \nu_0 = 1$ and $\deg_2 \nu_1 = 0$.*

Proof. Let Λ_0 be the set of those λ for which there exists an $x \in \partial M_0^\lambda$ with the following transversality property:

$$\text{If } y \in R_0^j(x) \text{ and } j > 0 \text{ then } V_j(y) \top \partial M_j^\lambda. \quad (3)$$

Since $F_i(T_i, \partial M_{i-1}^\lambda) \cap \partial M_i^\lambda = \emptyset$, it is clear that the set of pairs (λ, x) satisfying (3) is open. In particular, the set Λ_0 is open.

We will show that Λ_0 is dense. Take a point $z \in \partial M_0^\lambda$, choose ξ_1 close to λ_1 as a regular value of the map $g(t) = f_1(0)F_1(t, z)$ and put $S_1 = \{F_1(t, z) \mid$

$t \in g^{-1}(\xi_1)\}$. Having defined ξ_i and S_i (where $i < p$), choose ξ_{i+1} close to λ_{i+1} as a common regular value of the maps

$$g_y(t) = f_{i+1}(0)F_{i+1}(t, y)$$

for all $y \in S_i$, and put

$$S_{i+1} = \{F_{i+1}(t, y) \mid y \in S_i, t \in g_y^{-1}(\xi_{i+1})\}.$$

(Note that each S_i will be at most countable.) Condition (3) will be satisfied by the substitution $x = z$ and $\lambda = (\lambda_0, \xi_1 \dots \xi_p)$.

Suppose $\lambda \in \Lambda_0$ and take an $x \in \partial M_0^\lambda$ satisfying (3). By Morse-Sard theorem, we may take x to be a regular value of ν_0 . The number $\deg_2 \nu_0$ is equal to the reduction modulo 2 of the number of points in $\nu_0^{-1}(x)$. Consider any $y \in R_0^j(x)$, where $0 \leq j < p$. Since $y \notin M_{j+1}^\lambda$ and $F_{j+1}(T_{j+1}, y) \in \text{int} M_{j+1}^\lambda$, the transversality condition implies that the set

$$\{t \in (0, T_{j+1}) \mid F_{j+1}(t, y) \in \partial M_{j+1}^\lambda\}$$

has an odd number of elements, which proves $\nu_0^{-1}(x)$ has an odd number of points, so $\deg_2 \nu_0 = 1$.

Let Λ_1 be the set of those λ for which there exists an $x \in \partial M_p^\lambda$ with the following property:

$$\text{If } y \in R_p^j(x) \text{ and } j < p \text{ then } V_j(y) \top \partial M_j^\lambda. \quad (4)$$

As before, we first observe that Λ_1 is open and dense. Now suppose λ and $x \in \partial M_p^\lambda$ satisfy (4) and x is a regular value of ν_1 . Consider any $y \in R_p^\beta(x)$. Since $y \in M_\beta^\lambda$, $y \notin M_{\beta-1}^\lambda$. Since $y \notin \text{int} M_\beta^\lambda$ and $F^\beta(T_\beta, \beta)(M_{\beta-1}^\lambda) \subset \text{int} M_\beta^\lambda$, also $F_\beta^{-T_\beta}(y) \notin M_{\beta-1}^\lambda$. Thus, the transversality condition implies that the set

$$\{t \in (0, T_\beta) \mid F_\beta(-t, y) \in \partial M_{\beta-1}^\lambda\}$$

has an even number of elements. Now consider any $y \in R_p^{j+1}(x)$, where $\beta - 1 > j > 0$. Since ∂M_{j+1}^λ is connected and $F_{j+1}(-T_{j+1}, \partial M_{j+1}^\lambda) \cap \partial M_j^\lambda = \emptyset$, the set $F_{j+1}(-T_{j+1}, \partial M_{j+1}^\lambda)$ must be contained in $\text{int} M_j^\lambda$ or $M \setminus M_j^\lambda$. Similarly, ∂M_{j+1}^λ is contained in $\text{int} M_j^\lambda$ or $M \setminus M_j^\lambda$, since $\partial M_{j+1}^\lambda \cap \partial M_j^\lambda = \emptyset$. The transversality condition then implies that the set

$$\{t \in (0, T_{j+1}) \mid F_{j+1}(-t, y) \in \partial M_j^\lambda\}$$

has an even or odd number of elements independently of y . Thus we have proved that $\nu_1^{-1}(x)$ has an even number of points, which means that $\deg_2 \nu_1 = 0$. ■

Now, for any $\lambda \approx 0$ there is a diffeomorphic projection $\pi_\lambda : \partial M_p^\lambda \rightarrow \partial M_0$ close to the identity. It follows that the maps $f, g : N^\lambda \rightarrow \partial M_0$, where $f = \nu_0$ and $g = \pi_\lambda \circ \nu_1$ still satisfy $\deg_2(f) \neq \deg_2(g)$. The assumed coincidence property (mod 2) of the manifold ∂M_0 implies that $f(x_\lambda) = g(x_\lambda)$ for some point $x_\lambda \in N^\lambda$. Thus, for each value of $\lambda \approx 0$ there exists a trajectory which is nearly closed (up to the map π_λ). By letting $\lambda \rightarrow 0$ and selecting a suitable convergent subsequence, we obtain a p -periodic trajectory of the hybrid system \mathcal{S} . This completes the proof of Theorem 1. ■

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Heidelberg Berlin, 1976.

INSTYTUT MATEMATYKI
UNIwersytet Warszawski
UL. BANACHA 2
00-913 WARSZAWA, POLAND
E-mail address: witold@mimuw.edu.pl